Mixing time and diameter in random graphs

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Based on joint works with: Asaf Nachmias,
and Jian Ding, Eyal Lubetzky and Jeong-Han Kim.

Background

The mixing time of the lazy random walk on a graph G is

$$T_{\mathrm{mix}}(G) = T_{\mathrm{mix}}(G, 1/4) = \min\{t : \| \mathbf{p}^t(x, \cdot) - \pi(\cdot) \| \le 1/4 , \forall x \in V\} ,$$

where $\| \mu - \nu \| = \max_{A \subset V} |\mu(A) - \nu(A)|$ is the total variation distance.

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We are interested in the mixing time of the random walk on critical and near-critical random graphs.

The Erdos and Rényi random graph

The Erdos–Rényi random graph $G(n, p)$ is obtained from the complete graph on n vertices by retaining each edge with probability p and deleting it with probability $1 - p$, independently of all other edges. Let $\mathcal{C}_1$ denote the largest component of $G(n, p)$.

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Theorem (Erdos and Rényi, 1960)

If $p = \frac{c}{n}$ *then*

1. *If* $c < 1$ *then* $|\mathcal{C}_1| = O(\log n)$ *a.a.s.*
2. *If* $c > 1$ *then* $|\mathcal{C}_1| = \Theta(n)$ *a.a.s.*
3. *If* $c = 1$, *then* $|\mathcal{C}_1| \sim n^{2/3}$ *(proved later by Bollobas and Luczak)*

Random walk on the supercritical cluster in $G(n, p)$

Theorem (Fountoulakis and Reed & Benjamini, Kozma and Wormald)

If $p = \frac{c}{n}$ where $c > 1$, then the random walk on $\mathcal{C}_1$, the largest component of $G(n, p)$ (the unique component of linear size), has

$$T_{\mathrm{mix}}(\mathcal{C}_1) = \Theta(\log^2(n)).$$

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Question: [Benjamini, Kozma and Wormald] What is the order of the mixing time of the random walk on the largest component of the critical random graph $G(n, \frac{1}{n})$?

The diameter plays a crucial role

In bounded degree transitive graphs, T_{mix} is known to be at most $O(\mathrm{diam})^3$ and conjectured to be at most $O(\mathrm{diam})^2$. Note that diam$/2$ is always a lower bound. In the examples of random graphs we discuss, T_{mix} ranges between $O(\mathrm{diam})^3$ at criticality (average degree 1) and $O(\mathrm{diam})^2$ in the supercritical case (average degree > 1 but bounded).

Theorem

(Chung-Lu,...) Let $p = \frac{c}{n}$. Then

1. *If $c < 1$ then $diam(\mathcal{C}_1) = O(\sqrt{\log n})$ a.a.s., but there exists some other component of diameter $\Omega(\log n)$ (Luczak 1998).*
2. *If $c > 1$ then $diam(\mathcal{C}_1) = \Theta(\log n)$ a.a.s.*

More precise asymptotics in supercritical case by Fernoltz-Ramachandran (2007), Bollobás-Janson-Riordan (2007) culminating in recent work of Riordan-Wormald (2009).

Main Result- Critical case

Theorem (Nachmias, P. 2007)

Let $\mathcal{C}_1$ denote the largest connected component of $G(n, \frac{1}{n})$. Then for any $\epsilon > 0$ there exists $A = A(\epsilon) < \infty$ such that for all large n,

- $\mathbf{P}\Big(\mathrm{diam}(\mathcal{C}_1) \notin [A^{-1}n^{1/3}, An^{1/3}]\Big) < \epsilon\,,$
- $\mathbf{P}\Big(T_{\mathrm{mix}}(\mathcal{C}_1) \notin [A^{-1}n, An]\Big) < \epsilon\,.$

This answers the question of Benjamini, Kozma and Wormald.

Remark. This extends for p in the "critical window", i.e. $p = \frac{1+\lambda n^{-1/3}}{n}$.

An axiomatic approach using the intrinsic metric

For a vertex $v \in G$ let $\mathcal{C}(v)$ be the component containing v in G_p. Let $d_p(u, v)$ denote the distance between u and v in G_p. Define

$$B_p(v, k) = \{u \in \mathcal{C}(v) : d_p(v, u) \leq k\},$$

$$\partial B_p(v, k) = \{u \in \mathcal{C}(v) : d_p(v, u) = k\},$$

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We say that we have mean-field behavior if there exists $C > 0$ such that "at the critical p" for all $k > 0$

(i) $\mathbf{E}|\mathcal{E}(B_p(v, k))| \le Ck$,

(ii) $\mathbf{P}(|\partial B_p(v, k)| > 0) \le C/k$,

A general theorem

Theorem (Nachmias, P.)

Let G be a graph and $p \in (0,1)$ such that (i) and (ii) hold. (This is the case e.g. if G is d-regular and $p = 1/(d-1)$). Then

1. $\mathbf{P}\Big(\exists \mathcal{C} \text{ with } |\mathcal{E}(\mathcal{C})| > An^{2/3}\Big) < \epsilon\,,$
2. $\mathbf{P}\Big(\exists \mathcal{C} \text{ with } |\mathcal{C}| > \beta n^{2/3}\,,\ \mathrm{diam}(\mathcal{C}) \notin [A^{-1}n^{1/3}, An^{1/3}]\Big) < \epsilon\,,$
3. $\mathbf{P}\Big(\exists \mathcal{C} \text{ with } |\mathcal{C}| > \beta n^{2/3}\,,\ T_{\mathrm{mix}}(\mathcal{C}) \notin [A^{-1}n, An]\Big) < \epsilon\,.$

Wide range of underlying "high-dimensional" graphs

Thus, to conclude that the $\text{diam}(\mathcal{C}_1) \sim n^{1/3}$ and $T_{\text{mix}}(\mathcal{C}_1) \sim n$, one must show that at the chosen $p \in (0, 1)$ the conditions (i) and (ii) hold, and that $|\mathcal{C}_1| \sim n^{2/3}$. This was shown for:

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- Various tori: the Hamming hypercube $\{0,1\}^m$, Cartesian products of complete graphs K_n^m, high-dimensional torus $\mathbb{Z}_n^d$ where d is fixed but large, and $n \to \infty$ [Kozma and Nachmias 2009].

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Remark. The last result is proved for p in the mean-field scaling-window. Van der Hofstad and Heydenreich (2009) proves that $p_c(\mathbb{Z}^d)$ is in this scaling window.

Interpolating between critical and supercritical geometry?

How does the random graph $G(n, p)$ interpolate between the critical case $p = \frac{1}{n}$ and the supercritical case $p = \frac{1+c}{n}$?

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	Critical	$p=\frac{1+\epsilon}{n}$	Fully supercritical
Diameter	$n^{1/3}$	?	$\log n$
T_{mix}	n	?	$\log^2 n$

Very precise asymptotics for the Diameter obtained by Riordan-Wormald (2009) almost covering the supercritical regime except when $n\epsilon^3$ grows very slowly. After we obtained first-order asymptotics for the full regime, Riordan and Wormald extended their approach to cover it as well.

The diameter and mixing time in supercritical random graphs

Theorem (Ding, Kim, Lubetzky, P. 2009)

Let $\mathcal{C}_1$ *be the largest component of the random graph* $\mathcal{G}(n, p)$ *for* $p = (1+\epsilon)/n$*, where* $\epsilon^3 n \to \infty$ *and* $\epsilon = o(1)$*. Then w.h.p.,*

$$\mathrm{diam}(\mathcal{C}_1) = \frac{3 + o(1)}{\epsilon} \log(\epsilon^3 n)\,,$$

$$T_{\mathrm{mix}}(\mathcal{C}_1) = \Theta\big((1/\epsilon^3) \log^2(\epsilon^3 n)\big)\,.$$

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T_{mix}	n	$(1/\epsilon^3)\log^2(\epsilon^3 n)$	$\log^2 n$

Results for the giant component in $G(n, p)$

- Luczak (1991): Kernel is an almost cubic multigraph
- Pittel-Wormald (2005): Local CLT for $\mathcal{C}_1$ and the 2-core.
- Benjamini-Kozma-Wormald: $\mathcal{C}_1$ as an expander decorated by at most logarithmic trees.
- Our new result completely characterizes $\mathcal{C}_1$ via contiguity in the emerging supercritical case.

Anatomy of a young giant component in $G(n, p)$

Theorem (Ding, Kim, Lubetzky, P. 2009)

Let $\mathcal{C}_1$ be the largest component of the random graph $\mathcal{G}(n, p)$ for $p = \frac{1+\epsilon}{n}$, where $\epsilon^3 n \to \infty$ and $\epsilon = o(n^{-1/4})$. Then $\mathcal{C}_1$ is contiguous to the model $\tilde{\mathcal{C}}_1$, constructed in 3 steps as follows:

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1. *Let $Z \sim \mathcal{N}\left(\frac{2}{3}\epsilon^3 n, \epsilon^3 n\right)$, and select a random 3-regular (multi-)graph $\mathcal{K}$ on $N = 2\lfloor Z \rfloor$ vertices.*

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2. *Replace each edge of $\mathcal{K}$ by a path, where the path lengths are i.i.d.* Geom(ϵ).
3. *Attach an independent $\mathrm{Poisson}(1-\epsilon)$-Galton-Watson tree to each vertex.*

That is, $\mathbf{P}(\tilde{\mathcal{C}}_1 \in \mathcal{A}) \to 0$ implies $\mathbf{P}(\mathcal{C}_1 \in \mathcal{A}) \to 0$ for any set of graphs $\mathcal{A}$.

First-passage percolation and Open question

Results on diameter use first-passage percolation on (almost) 3-regular graphs, cf. recent work of Bhamidi, van-der Hofstad and Hoogemeistra.

Consider the high-dimensional torus $\mathbb{Z}_n^d$ with d fixed but large and $n \longrightarrow \infty$. Again, how does one interpolate between critical and supercritical geometry?

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	p_c	$p = (1+\epsilon)p_c$	Fully supercritical
Volume	$n^{2d/3}$	?	n^d
Diameter	$n^{d/3}$	?	n
T_{mix}	n^d	?	n^2

Upper bound on the diameter in the critical case

Recall that we define

$$B_p(v,k) = \{u \in \mathcal{C}(v) : d_p(v,u) \le k\},$$
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And assume that for all $k > 0$

(i) $\mathbf{E}|\mathcal{E}(B_p(v,k))| \le Ck\,,$

(ii) $\mathbf{P}(|\partial B_p(v,k)| > 0) \le C/k\,,$

Upper bound on the diameter in the critical case

If a vertex $v \in V$ satisfies $\mathrm{diam}(\mathcal{C}(v)) > R$, then $|\partial B_p(v, \lceil R/2 \rceil)| > 0$, thus by assumption (ii)

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Write

$$X = \Big|\{v \in V : |\mathcal{C}(v)| > M \text{ and } \mathrm{diam}(\mathcal{C}(v)) > R\}\Big|.$$

Then we have $\mathbf{E}X \leq \frac{2cn}{R}$. So we have

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$$\mathbf{P} \quad \Big(\exists \mathcal{C} \text{ with } |\mathcal{C}| > M \text{ and } \text{diam}(\mathcal{C}) > R\Big) \leq \mathbf{P}(X > M) \leq \frac{2cn}{MR},$$

and taking $M = \beta n^{2/3}$ and $R = A n^{1/3}$ concludes the proof.

Lower bound on the diameter

If $v \in V$ satisfies $\operatorname{diam}(\mathcal{C}(v)) \le r$ and $|\mathcal{C}(v)| > M$, then $|B_p(v, r)| > M$. Thus by assumption (i)

$$\mathbf{P}\Big(\operatorname{diam}(\mathcal{C}(v)) \le r \text{ and } |\mathcal{C}(v)| > M\Big) \le \frac{2r}{M}.$$

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Write

$$Y = \Big|\{v \in V : |\mathcal{C}(v)| > M \text{ and } \mathrm{diam}(\mathcal{C}(v)) < r\}\Big|\,.$$

Lower bound on the diameter (continued)

We learn that $\mathbf{E}Y \leq \frac{2rn}{M}$. As before this gives

$$\begin{aligned}&\mathbf{P}\left(\exists \mathcal{C} \in \mathbf{CO}(G_p) \text{ with } |\mathcal{C}| > M \text{ and } \mathrm{diam}(\mathcal{C}) < r\right) \\ \leq\ &\mathbf{P}(Y > M) \leq \frac{2rn}{M^2},\end{aligned}$$

and taking $M = \beta n^{2/3}$ and $r = A^{-1} n^{1/3}$ concludes the proof.

Upper bound on the mixing time

The upper bound $T_{\mathrm{mix}}(\mathcal{C}_1) \le O(n)$ follows from

Lemma

Let $G = (V, \mathcal{E})$ be a graph. Then the mixing time of a lazy simple random walk on G satisfies

$$T_{\mathrm{mix}}(G, 1/4) \le 8|\mathcal{E}(G)|\mathrm{diam}(G)\,.$$

The lower bound on the mixing time

Let $\mathcal{R}(u \leftrightarrow v)$ denote the effective resistance between u and v.

Lemma (Tetali 1991)

For a lazy simple random walk on a finite graph where each edge has unit conductance, we have

$$\mathbf{E}_v \tau_z = \sum_{u \in V} \deg(u) [\mathcal{R}(v \leftrightarrow z) + \mathcal{R}(z \leftrightarrow u) - \mathcal{R}(u \leftrightarrow v)] .$$

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Lemma (Nash-Williams 1959)

If $\{\Pi_j\}_{j=1}^J$ are disjoint cut-sets separating v from z in a graph with unit conductance for each edge, then the effective resistance from v to z satisfies

$$\mathcal{R}(v \leftrightarrow z) \geq \sum_{j=1}^{J} \frac{1}{|\Pi_j|} \,.$$